Analytic geometry and Euclidian space

Adam Boult (www.bou.lt)

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Analytic geometry

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Points, lines and affine transformations

1.1 Affine spaces

1.1.1 Lines

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Chapter 2

Euclidian transformations, lengths and angles

2.1 Linear metrics

2.1.1 Metrics

We defined a norm as:

$||v|| = v^T M v$

A metric is the distance between two vectors.

$d(u, v) = ||u - v|| = (u - v)^T M (u - v)$

Metric space

A set with a metric is a metric space.

2.1.2 Inducing a topology

Metric spaces can be used to induce a topology.

2.1.3 Translation symmetry

The distance between two vectors is:

$(v - w)^T M (v - w)$

So what operations can we do now?

As before, we can do the transformations which preserve $u^T M v$, such as the orthogonal group.

But we can also do other translations

$(v-w)^T M(v-w)$

$v^T Mv + w^T Mw - v^T Mw - w^T Mv$

so symmetry is now $O(3,1)$ and affine translations

Translation matrix

$[[1,x][0,1]]$ moves vector by x.

2.2 Specific groups

2.2.1 The affine group

2.2.2 The Euclidian group

2.2.3 The Galilean group

2.2.4 The Poincaré group

2.3 Non-linear norms

2.3.1 L_p norms (p-norms)

L^P norm

This generalises the Euclidian norm.

$||x||_p = (\sum_{i=1}^n |x|_i^p)^{1/p}$

This can defined for different values of p. Note that the absolute value of each element in the vector is used.

Note also that:

$||x||_2$

Is the Euclidian norm.

Taxicab norm

This is the L^1 norm. That is:

$||x||_1 = \sum_{i=1}^n |x|_i$

Angles

Cauchy-Schwarz

2.4 To linear forms

2.4.1 Norms

We can use norms to denote the "length" of a single vector.

$||v|| = \sqrt{\langle v, v\rangle}$

$||v|| = \sqrt{v^* M v}$

Euclidian norm

If $M = I$ we have the Euclidian norm.

$||v|| = \sqrt{v^* v}$

If we are using the real field this is:

$||v|| = \sqrt{\sum_{i=1}^{n} v_i^2}$

Pythagoras' theorem

If $n = 2$ we have in the real field we have:

$||v|| = \sqrt{v_1^2 + v_2^2}$

We call the two inputs x and y, and the length z.

$z = \sqrt{x^2 + y^2}$

$z^2 = x^2 + y^2$

2.4.2 Angles

Recap: Cauchy-Schwarz inequality

This states that:

$|\langle u, v\rangle|^2 \leq \langle u, u\rangle \dot{\langle} v, v\rangle$

Or:

$\langle v, u\rangle \langle u, v\rangle \leq \langle u, u\rangle \dot{\langle} v, v\rangle$

Introduction

$\langle v, u\rangle \langle u, v\rangle \leq \langle u, u\rangle \dot{\langle} v, v\rangle$

$\frac{\langle v, u\rangle \langle u, v\rangle}{||u||.||v||} \leq ||u||.||v||$

$$\frac{||u||.||v||}{\langle v, u\rangle} \geq \frac{\langle u, v\rangle}{||u||.||v||}$$

$$\cos(\theta) = \frac{\langle u, v\rangle}{||u||.||v||}$$

2.5 Other

2.5.1 Convex hulls

Chapter 3

Volumes, perimeters and surface areas

Chapter 4

2D polygons

4.1 Elementary geometry in 2 dimensions

4.1.1 Triangles

Area of a triangle

Circumference of a triangle

Sum of angles of a triangle

Angles in a triangle add to π.

4.1.2 Quadrilaterals

4.1.3 Oblongs

Area of an oblong

Circumference of an oblong

4.1.4 Squares

Area of a square

$A = l^2$

Circumference of a square

$C = 4l$

Angles in a square

Angles in a square sum to 2π.

4.1.5 Pentagon

4.2 Other

4.2.1 Border

4.2.2 Interior

4.2.3 Open

4.2.4 Closed

4.2.5 Self-intersecting polygon

Chapter 5

3D polygons

5.1 Elementary geometry in 3 dimensions

5.1.1 Pyramid

5.1.2 Cubes

Volume of a cube:

$V = l^3$

Surface area of a cube:

$A = 6l^2$

Chapter 6

Algebraic geometry and spheres

6.1 Circles

6.1.1 Defining circles

$x^2 + y^2 = r^2$

6.1.2 Area of a circle

$A = \pi r^2$

6.1.3 Circumference of a circle

$C = 2\pi r$

6.2 Spheres

6.2.1 Defining spheres

$x^2 + y^2 + z^2 = r^2$

6.2.2 Volume of a sphere

$V =$

6.2.3 Surface area of a sphere

$A =$

Preface

This is a live document, and is full of gaps, mistakes, typos etc.